# Making Beehives

## Student Objectives

Students answer their driving question and investigation questions through independent research, and then deliver this research to their classmates in a concise presentation form that includes a presentation website with video content. The student will:

***Mandatory (Level I Projects)***

- Understand which polygons can be used to tessellate a plane, and extend this reasoning from two dimensions to three dimensions.
- Set areas of a square, circle, equilateral triangle, and hexagon equal to each other for comparison of the ratio of perimeter to area.
- Determine the polygon that would most economically hold the greatest volume of liquid based on comparison of base area. Compute the theoretical volume.
- Using available materials construct a set prisms and a cylinder of whose bases are of equal area, and whose heights are equal.
- Determine experimentally the volume of each prism and cylinder using liquid, and compare these results to the theoretical values previously computed.
- Relate bees' food energy needs to their optimization of resources used to build honeycombs.

***Optional (Level II, III Projects Only)***

- Derive the formula for the area of a regular hexagon from the formula for the area of an equilateral triangle.
- Compute and compare the area of a circle to the area of a hexagon inscribed in it, and relate these areas to the volume of the cylinder and hexagonal prism.
- Compute and compare the area of the annulus formed by circles inscribed in and circumscribed about a hexagon, to the area of the hexagon.
- Describe and explain which transformations (rotation, translation, reflection, dilation) can be applied to produce a hexagonal tessellation.

## TEKS Objectives

This lab correlates to the following TEKS objectives:

(c)1 The student uses mathematical processes to acquire and demonstrate mathematical understanding. The student is expected to:

B. use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and reasonableness of the solution;

C. select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems;

D. communicate mathematical ideas, reasoning, and their implications using multiple representations, including symbols, diagrams, graphs, and language as appropriate;

(c)11 Two-dimensional and three-dimensional figures. The student uses the process skills in the application of formulas to determine measures of two- and three-dimensional figures. The student is expected to:

B. determine the area of composite two-dimensional figures comprised of a combination of triangles, parallelograms, trapezoids, kites, regular polygons, or sectors of circles to solve problems using appropriate units of measure;

C. apply the formulas for the total and lateral surface area of three-dimensional figures, including prisms, pyramids, cones, cylinders, spheres, and composite figures, to solve problems using appropriate units of measure; and

D. apply the formulas for the volume of three-dimensional figures, including prisms, pyramids, cones, cylinders, spheres, and composite figures, to solve problems using appropriate units of measure.

This lab correlates to the following Common Core State Standards:

HSG.GMD.A Explain volume formulas and use them to solve problems:

1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.

3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

HSG.GMD.B Visualize relationships between two-dimensional and three-dimensional objects:

4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

HSG.MG.A Apply geometric concepts in modeling situations:

1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

2 Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

## Project Background

People have observed for millennia that bees construct the cells of their honeycombs in the shape of hexagons. Anywhere in the world one looks inside a beehive, the same structure is found: regular, repeating patterns of hexagons. Bees seem to know instinctively to use this shape to build their honeycombs. Hexagons are one of three regular polygons that can tessellate, or tile, a flat region or plane. The angles formed by the sides of such polygons are a factor of 360, the number of degrees in one complete rotation.

Equilateral triangles and squares, whose sides form angles of 60 degrees and 90 degrees respectively, also tessellate a plane. Regular pentagons, on the other hand, do not tessellate because their sides form angles of 108 degrees, which is not a factor of 360. Three pentagons placed side to side so their vertices meet at one point complete 324 degrees of rotation, and adding another pentagon completes 432 degrees of rotation, with the consequence that a physical set of such pentagons could not lie flat on a surface.

The ability to tessellate is what enables hexagons to form a repeating pattern. The hexagon also encloses a maximum area for the length of its perimeter compared to squares or equilateral triangles. Circles maximize area for circumference, but do not tessellate (they leave gaps between successive circles even when closely packed). For bees building honeycombs, the hexagon strikes the best balance of volume enclosed by each cell for expenditure of resources to construct them.

In this project students will design, measure, and construct geometric solids that can hold liquid. They will use these containers to compare the volume of several shapes, holding base area and height of the containers constant.

Students will be able to use the formulas for the area of a square, a triangle, and a circle to complete the project. However, they will find the area formula for an equilateral triangle useful:

$$A_{\text{equi. triangle}} = \frac{s^2\sqrt{3}}{4} \qquad \text{Eq.1}$$

Students should discover that a regular hexagon is composed of six equilateral triangles, so Equation 1 can be modified as follows:

$$A_{\text{hexagon}} = \frac{3\sqrt{3}s^2}{2} \qquad \text{Eq.2}$$

## Materials and Special Preparation

### *Mandatory Equipment*

- Water-resistant material for making shapes (such as plastic-coated card stock)
- Scissors
- Tape
- Water
- Ruler or straightedge
- Compass
- Drawing paper, several sheets

### *Optional Equipment*

- Graduated cylinder
- Food coloring
- Pattern blocks
- Dynamic geometry software

The student handout for this activity lists the mandatory and optional equipment shown above that can be used as part of the student's project. Please review these items and the preparation notes for each prior to providing them to the student.

## Safety and Maintenance

Be certain that students add this important safety precaution to their normal classroom procedures:

- A construction project involves the use of sharp objects. Handle all sharp objects carefully, including a drawing compass, scissors or craft/hobby knives.

## Presentation Guidelines

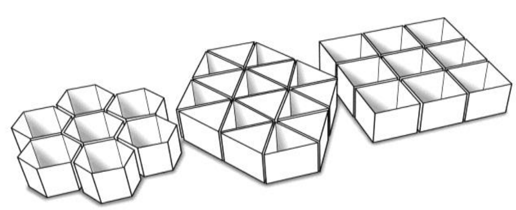
Use the following guidelines to assess the student's preparation, content knowledge, and presentation delivery.

### *Investigation Guidelines*

Follow these guidelines to assess each of the investigation questions from three parts of the student project handout. The Part 1 investigation is mandatory, while Parts 2 and 3 are optional. The third column indicates the difficulty level of the concept addressed in the investigation question.

| *PART 1 Investigation Questions (Mandatory)* | *Suitable Response* | *Difficulty* |
|---|---|---|
| 1. Which regular polygons can be used to tessellate, or tile, a plane, without overlapping or leaving gaps between them? | A square, an equilateral triangle, and a regular hexagon can each be used to tessellate a plane. | Easy |

| PART 1 Investigation Questions (Mandatory) | Suitable Response | Difficulty |
|---|---|---|
| 2. What three-dimensional solids can be formed from these polygons? How is a cylinder different from these three-dimensional solids? | A square forms a cube or rectangular prism if its height is greater then the length of a side. An equilateral triangle forms a triangular prism or a tetrahedron. A regular hexagon forms a hexagonal prism.<br>A cylinder has as its base a circle, which is not a polygon. The other solids are composed only of polygons. | Moderate |
| 3. Consider several types of three-dimensional solids, including hexagonal prisms, and cylinders. How can you compare the volume of one solid to another, both theoretically and experimentally? | Theoretically, the area formula for the base polygon can be used to compare the area of each solid. The height of each solid should be held constant. Then, the general volume formula for the solids, $$volume = area_{base} \times height$$ can be applied.<br>Experimentally, the solids' dimensions should be calculated and used to build hollow models into which liquid can be poured. The volume accepted by each model is compared by direct measurement. | Difficult |
| 4. Suppose you want to construct several two-dimensional figures that all have the same area, say 10 cm² for example. How can you use the area formula for a quadrilateral, an equilateral triangle, and a circle to determine the length of each side (or circumference) of the figures? | Solve each area formula for the dimension required, the diameter of the circle and one side of each of the regular polygons:<br>$diameter = 2\sqrt{\frac{a}{\pi}}$<br>$side_{\text{square}} = \sqrt{a}$<br>$side_{\text{eq. triangle}} = \sqrt{\frac{4a}{\sqrt{3}}}$<br>$side_{\text{hexagon}} = \frac{3\sqrt{3}\left(s^2\right)}{2}$<br>Then use 10 cm² for each area and compute the required dimension. | Moderate |

| PART 1 Investigation Questions (Mandatory) | Suitable Response | Difficulty |
|---|---|---|
| 5. What are the lengths of each side of a square and an equilateral triangle whose areas are both 10 $cm^2$? What is the length of the diameter of a circle whose area is 10 $cm^2$? | The dimensions are as follows:<br>$side_{\text{square}}$ = 3.162 cm<br>$side_{\text{eq. triangle}}$ = 4.806 cm<br><br>*diameter* = 3.568 cm | Easy |
| 6. What is the length of each side of a regular hexagon whose area is 10 $cm^2$ | For the hexagon, the length is:<br>$side_{\text{hexagon}}$ = 1.962 cm | Moderate |
| 7. Of all the two-dimensional figures whose areas and lengths of sides you computed, which figures, if any, have shorter perimeters? Is there a “shortest perimeter” figure that can also tessellate a plane? | The perimeters are:<br>Circumference of circle: 11.209 cm<br>Perimeter of square: 12.648 cm<br>Equilateral triangle: 14.417 cm<br>Perimeter of hexagon: 11.771 cm<br>The two shortest perimeters are the circle and the hexagon. Of these, only the hexagon can tessellate the plane. | Moderate |
| 8. How can you construct a set of hollow geometric solids, each of the same height and base area, using the least amount of material possible while still holding the largest volume of liquid possible? This is the same task that bees undertake when they construct their honeycombs from beeswax. Mathematically, which geometric solids are best suited for this task? | I used thin plastic recycled from a package, and packing tape, to build 3-D shapes. I chose a square prism, an equilateral triangular prism, and a hexagonal prism because they are the shapes that can tile a plane, or tessellate. I chose a cylinder because it has the largest volume even though it can’t tessellate a plane.<br>I made each shape so the base area was 10 $cm^2$ and the height of each shape was 10 cm.<br> | Difficult |

| PART 1 Investigation Questions (Mandatory) | Suitable Response | Difficulty |
|---|---|---|
| 9. Biologists have shown that a honeybee needs to eat between six and eight grams of honey in order to produce one gram of beeswax. In terms of bees' food energy requirements, discuss why it is important for bees to use beeswax as efficiently as possible. Why do bees build their honeycombs using the shapes they do? | Bees build their honeycomb cells this way because the more wax they need to use, the more they need to eat to have the energy to produce that amount of wax.<br>By using hexagons to build the cells they minimize materials used to build, while maximizing the volume of honey each cell can hold. | Easy |

| PART 2 Investigation Questions (Optional) | Suitable Response | Difficulty |
|---|---|---|
| 1. How can the formula for the area of a regular hexagon be derived from the formula for the area of an equilateral triangle? | Formula for area of an equilateral triangle:<br>$A = \frac{s^2\sqrt{3}}{4}$<br>A regular hexagon is composed of 6 equilateral triangles. Multiply by 6 the formula for the equilateral triangle:<br>$A_{\text{hex}} = 6A_{\text{eq. tri.}} = 6\frac{s^2\sqrt{3}}{4}$<br>$A_{\text{hex}} = \frac{3\sqrt{3}\left(s^2\right)}{2}$ | Moderate |

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| 2. Suppose a certain species of honeybee made its honeycomb with circular cells packed as closely together as possible (hexagonal packing). What volume of beeswax would be necessary to fill in all the gaps between the one cell and the other six cells surrounding it? Since you don't know the size of this species of bee (and therefore the size of its honeycomb cells), express this volume in terms of the ratio of the area of a circle to the area of a hexagon inscribed in that circle? | If the hexagon is inscribed in a circle, then each vertex of the hexagon lies on the circumference of the circle. Therefore, the radius of the circle is also the length of the side of one of the equilateral triangles that composes the hexagon.<br>The area of the circle is:<br>$A = \pi r^2$<br>The area of the inscribed hexagon is expressed in terms of the circle's radius:<br>$A = \frac{3\sqrt{3}\left(r^2\right)}{2}$<br>The ratio of the area of the circle to the area of the hexagon is:<br>$\frac{\pi r^2}{\frac{3\sqrt{3}\left(r^2\right)}{2}}$<br>$= \frac{\pi r^2}{1} \cdot \frac{2}{3\sqrt{3}\left(r^2\right)}$<br>$= \frac{2\pi}{3\sqrt{3}}$<br>$\approx 1.209$ | Difficult |

| PART 3 Investigation Questions (optional) | Suitable Response | Difficulty |
|---|---|---|
| 1. Consider a hexagon that has a circle inscribed in it and a second circle circumscribed about it. The annulus formed by the two concentric circles has an area *A*, while the hexagon has an area *A’*. What percent of *A’* is *A*? | The area of the annulus is given by: $$A_{annulus} = \pi(R^2 - r^2)$$ Where $R$ is the radius of the larger circle and $r$ is the radius of the smaller circle, and $$r = \frac{R\sqrt{3}}{2}$$ Therefore $$A_{annulus} = \pi\left(R^2 - \left(\frac{R\sqrt{3}}{2}\right)^2\right)$$ $$A_{annulus} = \pi\left(R^2 - \left(\frac{3R^2}{4}\right)\right)$$ $$A_{annulus} = \pi R^2\left(1 - \frac{3}{4}\right)$$ $$A_{annulus} = \frac{\pi R^2}{4}$$ Hence $$\frac{A}{A'} = \frac{\frac{\pi R^2}{4}}{\frac{3\sqrt{3}R^2}{2}}$$ $$\frac{A}{A'} = \frac{\pi R^2}{4} \cdot \frac{2}{3\sqrt{3}R^2}$$ $$\frac{A}{A'} = \frac{\pi}{6\sqrt{3}}$$ $$\frac{A}{A'} = \frac{\pi\sqrt{3}}{18}$$ $$\frac{A}{A'} \approx 0.3025$$ $$\frac{A}{A'} \approx 30\%$$ The annulus has an area about 30% that of the hexagon with inscribed and circumscribed circles. | Difficult |

| | | |
|---|---|---|
| 2. Of the transformations mathematically possible, which could be applied to the hexagonal honeycomb pattern of beehives? | The transformations that could be applied to hexagonal honeycomb patterns are rotation, translation, and reflection. One cell can be rotated 120° and it is still congruent to all other cells. A cell can be translated by a distance equal to its dimensions and is still congruent to all other cells. A cell can be reflected about any of its segments and is still congruent to all other cells.<br><br>A cell cannot be transformed by dilation because although it would still be similar, it would no longer be congruent to the other cells. | Moderate |

### *Presentation Delivery Guidelines*

Follow these guidelines to assess the student's preparation and presentation delivery to the class. To receive excellent marks, students have:

| | |
|---|---|
| ***Preparation:*** | ♦ clearly invested at least 3–4 hours doing topical research<br>♦ brought and assembled all equipment necessary for their presentation<br>♦ prepared a supporting handout for their presentation<br>♦ prepared a thorough presentation discussion for their classmates |
| ***Delivery:*** | ♦ delivered their presentation content clearly and thoroughly, not speaking so quickly that the student audience cannot follow<br>♦ integrated their research and experimental setup into the presentation as visual support<br>♦ answered all question posed by their classmates to the best of their ability<br>♦ answered all question posed by their classmates correctly and thoughtfully |
| ***Website:*** | ♦ prepared a website explaining their project and the driving question associated with it<br>♦ included a thorough, well-written, explanation of the answer to their driving question and the theory behind the over-arching concept<br>♦ included supporting pictures, videos, and research references (including web-links) |
| ***Video:*** | ♦ recorded a supporting video for their website<br>♦ edited the supporting video in an attempt to produce a good–excellent quality video<br>♦ included a thorough, well-spoken, explanation of the answer to their driving question and the theory behind the over-arching concept<br>♦ included footage of their experimental setup with an explanation |
| ***Handout:*** | ♦ prepared a supporting handout for their presentation<br>♦ included a brief, but well-written, explanation of the answer to their driving question, the theory behind the over-arching concept, and its application to real life<br>♦ included pictures of their experimental setup with explanation<br>♦ included references to all web-links and to the presentation website URL (or QR code) |

## Synthesis Questions

Below are sample responses to the questions in the Synthesis Questions section in the student handout.

### 1. Why do only certain kinds of polygons tessellate?

Polygons tessellate only if they have angles that are a factor of 360. For example, a square tessellates because its angles are each 90°, so that 4 successive rotations of a square complete 360°. An equilateral triangle must complete 6 successive rotations because its angles are each 60°. The rotations can be seen in a tessellation where the vertices of the polygons all meet at one point. A regular pentagon will not tessellate because its angles are each 108°, which is not a factor of 360 (360/108 = 3.33). The pentagons would leave a gap where their vertices meet, or adding another pentagon would result in an overlap.

### 2. What are some other examples of how hexagons are used in nature?

The Giants' Causeway in Northern Ireland has hexagonal formations of igneous rock. Snowflakes form in hexagonal patterns. Atoms of many kinds, including carbon, link together in hexagonal structures to form molecules. Certain types of crystals form hexagonal structures.

### 3. How are hexagons used by people in daily life? Why are hexagons chosen as the shape in these examples?

Hexagons are used by people in everything from quilt and knitting patterns to the structure of super-strong composite materials used to manufacture lightweight structures, such as airplanes. Soccer balls are made partially of hexagons (and also pentagons). They aren't the only shape used, because they lie flat (tile a plane) and a ball needs to be spherical. Hexagons are used because they form patterns that can be beautiful in the artistic sense, such as the tessellations that decorate buildings and outdoor spaces. Hexagons are efficient at optimizing volume for surface area or perimeter. Hexagons used in structures also do not shear as easily as a network of triangles or squares, because they do not have any straight lines running through the network.

## Extension and Real Life Application

Below are sample responses to the questions in the Extension and Real Life Application section in the student handout.

### 1. Discuss at least one real life application (and extension) of concepts relevant to this entire project, in addition to what you have discussed in the previous question(s).

Holding the greatest volume for the least material to make the container is important in any situation where economy or efficiency is a goal. This is also true in two dimensions as well, for example when trying to cut as many flat pieces of something from one sheet of material. Computer software and laser cutting are often used to be more efficient at cutting metal parts, clothing fabric, or upholstery material from the main sheet of metal, cloth, or leather. Cooks also use a technique of optimizing to get the most raviolis, pot-stickers, biscuits, or cookies cut out of a piece of dough or pasta.

### 2. Scientists estimate that one-third of food consumed by humans is dependent on insect pollination. Bees carry out the majority of this pollination. Discuss the implications for the world food supply if bees suffer some kind of crisis. Is there any such crisis currently affecting bees?

Our food supply would suffer from scarcities of many items, which would drive up the cost beyond the reach of many people, especially in developing countries. Since food is a necessity, like water and shelter, people might go to war against one another to protect what they have.

Bees around the world currently are suffering from colony collapse disorder, in which the bees abandon their hives for reasons that are not clear to scientists. There is a shortage of pollinator bees as a result, both in the United States and other countries around the world.

## 3. What is "closest packing" and how does it relate to the chemical structures of various crystalline substances?

Closest packing in crystalline structures means that the atoms, ions, or molecules are arranged in the way that makes the most use of the space. The atoms or molecules can be modeled as spheres, which are closest packed in a hexagonal arrangement or a cubic arrangement. In chemistry the structure is described with the particles on the lattice points, or intersections of imaginary lines that connect the particles' centers. Chemists use these models to explain and predict the bonding and structural behavior or properties of different crystalline materials.

| Mathematics Investigation Report – Assessment Rubric<br>Level II/III | | Name:<br>Math Class: | | | | |
|---|---|---|---|---|---|---|
| **Component** | To receive highest marks the student: | **4<br>Expert** | **3<br>Prac-<br>titioner** | **2<br>Appren-<br>tice** | **1<br>Novice** | **0<br>No<br>Attempt** |
| 1. Preparation and Research | ☐ Clearly invested several hours doing relevant research<br>☐ Brought and prepared all items necessary for their presentation<br>☐ Prepared supporting handout for their presentation<br>☐ Prepared a thorough presentation discussion for their classmates | | | | | |
| 2. Demonstrations, Models, or Experiments | ☐ Obtained all necessary materials and used them to thoroughly explore the guiding questions for Part 1 of the investigation<br>☐ Completed and documented results, data, and observations of any guiding questions investigated in Parts 1 – 3 of the investigation<br>☐ Performed experiments or demonstrations proficiently for others, or explained clearly a model and the concepts the student investigated with the model | | | | | |
| 3. Content | ☐ Presented written and spoken explanations that were mathematically accurate and paraphrased in the student's own words<br>☐ Answered all questions posed by their teacher or classmates correctly and thoughtfully<br>☐ Answered project synthesis and real-world application questions correctly and thoroughly<br>☐ Communicated clearly an understanding of the connection between their model or experiments and the driving question and theory behind the over-arching concept | | | | | |
| 4. Technology | ☐ Prepared a **website** explaining their project and the driving question associated with it, including:<br>• A thorough, well-written explanation of the answer to the driving question and the theory behind the over-arching concept<br>• Supporting pictures, videos, and research references (including web-links)<br>☐ Recorded a supporting **video** for the website<br>• Edited the supporting video in an attempt to produce a good–excellent quality video<br>• Included footage of their experimental setup with an explanation<br>• Included a thorough, well-spoken explanation of the answer to the driving question and the theory behind the over-arching concept<br>☐ Produced a project **brochure** with 2 **QR codes** linking to video and to website | | | | | |
| 5. Interdisciplinary Connections | ☐ Chose and completed ELA option to best illustrate the project's objectives and monitor student progress<br>☐ Included the ELA component on project website as separate page with appropriate design<br>☐ Chose a related SS connection based on consultation with SS teacher<br>☐ Included the SS connection on project website as separate page with appropriate design<br>☐ Included ELA component and SS connection in video presentation | | | | | |
| 6. Real-World Application and Extension | ☐ Identified in written and spoken explanations the application of the topic to the real world, including specific examples<br>☐ Thoroughly discussed relevance of the topic to real life | | | | | |
| 7. Presentation | ☐ Delivered their content clearly and thoroughly, in an organized, logical manner<br>☐ Integrated their research and experimental setup into the presentation as visual support | | | | | |
| | **Total Points for Investigation (Maximum of 24 Points)** | | | | | |

Guidelines for Marks:

4 = **Expert:** Distinguished command of the topic; students show insightful and sophisticated communication of their understanding

3 = **Practitioner:** Strong command of the topic; students show reasonable and purposeful communication of their understanding

2 = **Apprentice:** Moderate command of the topic; students show adequate but basic communication of their understanding

1 = **Novice:** Partial command of the topic; students show limited and insufficient communication of their understanding

0 = **No Attempt**